Uniform Continuity

First of all, let's recall the definitions of different types of continuity and compare them.

Definition (c.f. Definition 5.4.1)**.** Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ be a function.

(a) f is said to be *continuous at* $x \in A$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(u)| < \varepsilon \quad \text{whenever } u \in A \text{ and } |x - u| < \delta.$$

(b) f is said to be *continuous on* A if f is continuous at every $x \in A$. i.e., for any $x \in A$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(u)| < \varepsilon \quad \text{whenever } u \in A \text{ and } |x - u| < \delta.$$

(c) f is said to be *uniformly continuous on* A if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(u)| < \varepsilon \qquad \text{whenever } x, u \in A \text{ and } |x - u| < \delta.$$

Remark. Notice that the uniform continuity on a set implies the continuity on the same set. i.e., uniform continuity is a stronger continuity. Try to prove it!

Example 1. The function $f(x) = 1/(1 + x^2)$ is uniformly continuous on $\mathbb{R}$.

Proof. Note that for any $x, u \in \mathbb{R}$,

$$|f(x) - f(u)| = \left| \frac{1}{1 + x^2} - \frac{1}{1 + u^2} \right| = \frac{|x + u|}{(1 + x^2)(1 + u^2)} |x - u|.$$

Hence we estimate:

$$\frac{|x + u|}{(1 + x^2)(1 + u^2)} \leq \frac{|x| + |u|}{(1 + x^2)(1 + u^2)} \leq \frac{|x|}{1 + x^2} + \frac{|u|}{1 + u^2} \leq \frac{1}{2} + \frac{1}{2} = 1.$$

Then for any $\varepsilon > 0$, we can take $\delta = \varepsilon$. Then whenever $x, u \in \mathbb{R}$ and $|x - u| < \delta$,

$$|f(x) - f(u)| \leq |x - u| < \delta = \varepsilon.$$

The result follows by definition. □

Example 2. The function $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

Proof. We need to choose an $\varepsilon > 0$ and show that for every $\delta > 0$, there are two real numbers x and u such that

$$|x - u| < \delta \quad \text{and} \quad |f(x) - f(u)| \geq \varepsilon.$$

Take $\varepsilon = 1$. For every $\delta > 0$, take $x = 1/\delta + \delta/2$ and $u = 1/\delta$. Then $|x - u| < \delta$ and

$$|f(x) - f(u)| = \left| \left(\frac{1}{\delta} + \frac{\delta}{2} \right)^2 - \left(\frac{1}{\delta} \right)^2 \right| = 1 + \frac{\delta^2}{4} \geq 1.$$

The result follows by definition. □

There are helpful ways to establish uniform continuity. In addition to continuity, if further conditions on the domain of definition or the function are imposed, we may get uniform continuity.

Uniform Continuity Theorem (c.f. 5.4.3)**.** *Let I be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f is uniformly continuous on I.*

The following class of functions are called **Lipschitz functions**. They are uniformly continuous. Thus, it gives another way for us to establish uniform continuity.

Definition (c.f. Definition 5.4.4)**.** Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ be a function. f is said to be a *Lipschitz function* or said to *satisfy a Lipschitz condition* on A if there exists a constant $K > 0$ such that

$$|f(x) - f(u)| \leq K|x - u|, \quad \forall x, u \in A.$$

Theorem (c.f. Theorem 5.4.5)**.** *A Lipschitz function on A is uniformly continuous on A.*

Remark. The function in **Example 1** is a Lipschitz function with constant $K = 1$.

Example 3. The function $f(x) = \sqrt{x}$ is uniformly continous on $[0, 1]$ but does not satisfy a Lipschitz condition on $[0, 1]$.

Proof. Since f is continuous on the closed bounded interval $[0, 1]$, it is uniformly continuous on $[0, 1]$ by the **Uniform Continuity Theorem**. It remains to show that f does not satisfy a Lipschitz condition on $[0, 1]$. i.e., for any $K > 0$, there exists $x, u \in [0, 1]$ such that

$$|f(x) - f(u)| = |\sqrt{x} - \sqrt{u}| > K|x - u|.$$

This can be done by taking $x = 1/4K^2$ and $u = 0$. Then

$$|\sqrt{x} - \sqrt{u}| = \frac{1}{2K} \quad \text{and} \quad K|x - u| = \frac{1}{4K}.$$

The result follows. □

Nonuniform Continuity Criteria (c.f. 5.4.2)**.** *Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. TFAE:*

(i) f is not uniformly continuous on A.

(ii) There exists $\varepsilon_0 > 0$ such that for all $\delta > 0$, there exists $x_\delta, u_\delta \in A$ such that

$$|x_\delta - u_\delta| < \delta \quad \textit{and} \quad |f(x_\delta) - f(u_\delta)| \geq \varepsilon_0.$$

(iii) There exists $\varepsilon_0 > 0$ and two sequences (x_n) and (u_n) in A such that

$$\lim(x_n - u_n) = 0 \quad \textit{and} \quad |f(x_n) - f(u_n)| \geq \varepsilon_0, \quad \forall n \in \mathbb{N}.$$

Theorem (c.f. Theorem 5.4.7)**.** *If $f : A \to \mathbb{R}$ is uniformly continuous on A and if (x_n) is a Cauchy sequence in A, then $(f(x_n))$ is a Cauchy sequence in $\mathbb{R}$.*

Example 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose

$$\lim_{x\to-\infty} f(x) = L = \lim_{x\to\infty} f(x).$$

Show that f is uniformly continuous on $\mathbb{R}$.

Solution. Let $\varepsilon > 0$. By definition of limits to ∞, there exists $a < 0$ and $b > 0$ such that

$$|f(x) - L| < \frac{\varepsilon}{2} \quad \text{whenever } x < a \text{ or } x > b.$$

i.e., $|f(x) - f(u)| \le |f(x) - L| + |f(u) - L| < \varepsilon$ whenever $x, u \in (-\infty, a) \cup (b, \infty)$. Now, notice that f is continuous on the closed bounded interval $[a - 1, b + 1]$. By the **Uniform Continuity Theorem**, there exists $\delta' > 0$ such that

$$|f(x) - f(u)| < \varepsilon, \quad \text{whenever } |x - u| < \delta' \text{ and } x, u \in [a - 1, b + 1].$$

Take $\delta = \min\{\delta', 1\}$. Then whenever $|x - u| < \delta$, either

$$x, u \in [a - 1, b + 1] \quad \text{or} \quad x, u \in (-\infty, a) \cup (b, \infty).$$

Hence in both cases, we have

$$|f(x) - f(u)| < \varepsilon.$$

It follows that f is uniformly continuous on $\mathbb{R}$.

Example 5 (c.f. Section 5.4, Ex.12)**.** Show that if f is continuous on $[0, \infty)$ and uniformly continuous on $[a, \infty)$, for some positive constant a, then f is uniformly continuous on $[0, \infty)$.

Solution. Let $\varepsilon > 0$. Since f is uniformly continuous on $[a, \infty)$, there exists $\delta_1 > 0$ such that whenever $x, u \in [a, \infty)$ and $|x - u| < \delta_1$,

$$|f(x) - f(u)| < \varepsilon.$$

By **Uniform Continuity Theorem**, f is uniformly continuous on $[0, a + \delta_1]$. Hence there exists $\delta_2 > 0$ such that whenever $x, u \in [0, a + \delta_1]$ and $|x - u| < \delta_2$,

$$|f(x) - f(u)| < \varepsilon.$$

Take $\delta = \min\{\delta_1, \delta_2\}$. Then whenever $x, u \in [0, \infty)$ and $|x - u| < \delta$, either $x, u \in [0, a + \delta_1]$ or $x, u \in [a, \infty)$. Hence in both cases, we have

$$|f(x) - f(u)| < \varepsilon.$$

Thus f is uniformly continuous on $[0, \infty)$.